FB Mathematik
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Exercises for the course "Complex Analysis"

Exercise 1: a) Prepare a lecture on summability (on 08.05.2006);
b) Prepare a lecture on the "Umordnungssatz" (on 08.05.2006).

Exercise 2: a) Let $A = \{(z_1, z_2) \in \mathbb{C}^2 |\ z_2 = 0\}$ be the z_1-axis. Show that any holomorphic function f on $\mathbb{C}^2 \setminus A$ can be developed into a "Laurent" series

$$f(z) = \sum_{\substack{\nu_1 \geqslant 0 \\ \nu_2 \in \mathbb{Z}}} a_{\nu_1 \nu_1} z_1^{\nu_1} z_2^{\nu_2}$$

and find a formula for the coefficients.

b) Prove in addition that the family obtained, $(a_{\nu_1 \nu_1} z_1^{\nu_1} z_2^{\nu_2})$, is locally uniformly summable.

Exercise 3: Determine the domains of convergence $D(f)$ for the power series f in two variables

$$f = \sum_{n \geqslant 0} xy^n, \quad f = \sum_{n \geqslant 0} (xy)^n, \quad f = \sum_{\nu, \mu \geqslant 0} \frac{\nu}{\mu!} z^\nu w^\mu.$$

Exercise 4: Let $\Omega \subset \mathbb{C}^n$ be an open set and let $\Omega \xrightarrow{f} \mathbb{C}^n$ be a C^1-map. Verify the formula

$$\det J(f) = \det \begin{pmatrix} (\frac{\partial f_\nu}{\partial z_\mu})_{\nu,\mu} & (\frac{\partial f_\nu}{\partial \bar{z}_\mu})_{\nu,\mu} \\ & \\ (\frac{\partial \bar{f}_\nu}{\partial z_\mu})_{\nu,\mu} & (\frac{\partial \bar{f}_\nu}{\partial \bar{z}_\mu})_{\nu,\mu} \end{pmatrix}.$$

Exercise 5: (1) Let $G \subset \mathbb{C}^2$ be a domain. Let $f \neq 0$ be a holomorphic function on G such that $f(a) = 0$ for some $a \in G$, $a = (a_1, a_2)$. Assume

$\frac{\partial f}{\partial z_2}(a) \neq 0$. Show that there are open neighbourhoods $U_1(a_1),\ U_2(a_2) \subset \mathbb{C}$ and a holomorphic function g on U_1 such that

$$\{(z_1, z_2) \in U_1 \times U_2 |\ f(z_1, z_2) = 0\} = \{(z, g(z))|\ z \in U_1\}.$$

Hint: *use the local inversion theorem for the mapping*

$$(z_1, z_2) \mapsto (z_1, f(z_1, z_2)).$$

(2) Indicate the generalization of (1) for domains $G \subset \mathbb{C}^m \times \mathbb{C}^n$ and holomorphic mappings $G \xrightarrow{f} \mathbb{C}^n$ with a sketch of the proof.

Exercise 6: (1) Let $G \subset \mathbb{C}^n$ be a domain, let $f \not\equiv 0$ be a holomorphic function on G, and let

$$\{z \in G|\ f(z) = 0\}$$

be the zero set of f. Show that the interior $\overset{\circ}{Z}$ is empty.

(2) Show that for any $a \in Z$ there is a complex line L through a such that a is an isolated point of $L \cap Z$ (there exists $U(a)$ such that $L \cap Z \cap U = \{a\}$). A complex line through a should be the image of an affine map $\mathbb{C} \to \mathbb{C}^n$,

$$z \mapsto (a_1 + \lambda_1 z, \ldots, a_n + \lambda_n z).$$

(3) Choose a complex line as in (2). Then after a linear change of coordinates there are coordinates $z_1, \ldots, z_n$ of $\mathbb{C}^n$ such that

$$L = \{z_1 = a_1, \ldots, z_{n-1} = a_{n-1}\},$$

and there are radii $0 < r_1, \ldots, r_n$, $0 < \rho_n < r_n$ such that

$$\{z \in \mathbb{C}^n|\ |z_\nu - a_\nu| < r_\nu \text{ for } 1 \leqslant \nu \leqslant n-1, \rho < |z_n - a_n| < r_n\}$$

does not meet $Z \cap P_r(a)$. **Draw a figure!**

(4) Show, using (3), that any bounded holomorphic function g on $P_r \setminus Z$ can be holomorphically extended to the whole of P_r.

(5) (*First Riemann extension theorem*) Let g be a holomorphic function on $G \setminus Z$ and assume that g is bounded near Z (for all $a \in Z$ there exists a neighbourhood $U(a)$ such that $g|_{U \setminus Z}$ is bounded). Then g can be holomorphically extended across Z.

Definition. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be complex manifolds. A continuous map $X \xrightarrow{f} Y$ is called holomorphic if for any charts $(U, \varphi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ the map (also called local expression of f) $\psi \circ f \circ \varphi^{-1} : \varphi(U \cap f^{-1}(V)) \to \psi(V)$ is holomorphic.

Exercise 7: Let $\hat{\mathbb{C}} \xrightarrow{f} \hat{\mathbb{C}}$ be one of the maps

$$f(z) = \begin{cases} 1/z & z \neq 0, \infty \\ 0 & z = \infty \\ \infty & z = 0 \end{cases} \quad , \quad f(z) = \begin{cases} z + a & z \neq \infty \\ \infty & z = \infty \end{cases} .$$

Show that f is holomorphic and has a holomorphic inverse.

Exercise 8: Let $X \subset \mathbb{C}$ be a domain and let f be a meromorphic function on X (i. e., there exists a discrete set $A \subset X$ such that f is holomorphic on $X \setminus A$ and has a pole at each $a \in A$). Let $X \xrightarrow{\hat{f}} \hat{\mathbb{C}}$ be defined by

$$\hat{f}(x) = \begin{cases} f(x) & \text{if } x \notin A \\ \infty & \text{if } x \in A. \end{cases}$$

Show that $\hat{f}$ is a holomorphic map.

Exercise 9: Let $X \xrightarrow{f} Y$ and $X \xrightarrow{g} Y$ be holomorphic maps between complex manifolds and assume that f and g coincide on some non-empty open subset of X. Prove that $f = g$.

Exercise 10: Let $V \xrightarrow{A} V$ be a linear isomorphism of the $\mathbb{C}$ - vector space V, $\dim V = n + 1$. Verify that the map $\langle v \rangle \mapsto \langle A(v) \rangle$ is a holomorphic map from $\mathbb{P}(V)$ to $\mathbb{P}(V)$.

Exercise 11: Let $H \subset \mathbb{C}^{n+1}$ be any n - dimensional linear subspace with equation $a(z) = a_0 z_0 + \cdots + a_n z_n = 0$, assume that $a_i \neq 0$ and let

$$U = \mathbb{P}_n(\mathbb{C}) \setminus \mathbb{P}(H) \xrightarrow{\varphi} \mathbb{C}^n$$

be defined by

$$\langle x \rangle \mapsto \left(\frac{x_0}{a(x)}, \ldots, \frac{\widehat{x_i}}{a(x)}, \ldots, \frac{x_n}{a(x)} \right).$$

Verify that (U, φ) belongs to the completion of the atlas

$$\mathcal{A} = \{(U_0, \varphi_0), \ldots, (U_n, \varphi_n)\}$$

as defined in the lectures.

Exercise 12: Let $C \subset \mathbb{P}_2(\mathbb{C})$ be a conic (defined by a homogeneous polynomial of degree 2). Then any line L (defined by a linear form) meets C in 2 distinct points or is "tangent" to C.
Hint: consider affine charts as neighbourhoods of the intersection points.

Definition. Let $D \subset \mathbb{C}^n$ be a domain, let $f_1, \ldots, f_k \in \mathcal{O}(D)$ be holomorphic functions and let

$$A := Z(f_1, \ldots, f_k) = \{x \in D | \; f_1(x) = \cdots = f_k(x) = 0\}$$

be the common zero locus. A is called an analytic set. A is called smooth (non-singular) at $a \in A$, or a is a smooth point of A, if there is an open neighbourhood $U(a) \subset D$ such that $A \cap U$ is a submanifold of U. The singular locus of A is the complement of the set of smooth points of A.

Exercise 13: Determine the singular locus of $Z(f) \subset \mathbb{C}^2$ for $f = z_1^3 - z_2^2$ and $f = z_1^2 - z_2^2 - z_2^3$ and try to draw a picture of $Z(f)$ near the origin.

Exercise 14: Let f and g be the holomorphic functions on $\mathbb{C}^4$ given by $f = z_1^3 + z_2^3 + z_3^3 + z_4^3$ and $g = z_3^2 - z_1 z_2$.
a) Determine the singular locus of $Z(f)$ and $Z(g)$.
b) Determine the singular locus of $Z(f, g)$.

Exercise 15: Let $C \subset \mathbb{P}_3(\mathbb{C})$ be the set of points $\langle z_0, z_1, z_2, z_3 \rangle$ with

$$\operatorname{rank} \begin{pmatrix} z_0 & z_1 & z_2 \\ z_1 & z_2 & z_3 \end{pmatrix} \leqslant 1.$$

a) Show that C is well defined by this condition and that C is a smooth curve, i. e., a submanifold of dimension 1 in $\mathbb{P}_3(\mathbb{C})$.
b) Can C be biholomorphic to $\mathbb{P}_1(\mathbb{C})$ as a complex manifold? If yes, present a biholomorphic map.
Exercise 16: Let $f_{d+1}, \ldots, f_n \in \mathbb{C}[z_0, \ldots, z_n]$ be homogeneous polynomials (of possibly different degrees), $0 < d \leqslant n$, and let

$$A = Z(f_{d+1}, \ldots, f_n) = \{\langle x \rangle | \; f_{d+1} = \cdots = f_n = 0\}.$$

Assume that rank $\left(\frac{\partial f_\mu}{\partial z_\nu}(a)\right) = n - d$ for any $\langle a \rangle \in A$. Verify that A is a submanifold of $\mathbb{P}_n(\mathbb{C})$ of dimension d.

Exercise 17: (Rational normal curve in $\mathbb{P}_n(\mathbb{C})$). Let $\mathbb{P}_1(\mathbb{C}) \xrightarrow{f} \mathbb{P}_n(\mathbb{C})$ be defined by

$$f(\langle t_0, t_1 \rangle) = \langle t_0^n, t_0^{n-1} t_1, \ldots, t_1^n \rangle.$$

Show that

(1) f is a well-defined, injective, holomorphic map;

(2) $C = \mathrm{Im}(f)$ is a 1-dimensional submanifold (smooth curve);

(3) $\mathbb{P}_1(\mathbb{C}) \xrightarrow{f} C$ is biholomorphic;

(4) Try to find homogeneous polynomials whose common zero set is C (generalizing the case $n = 2$).

Exercise 18: (Quadrics in $\mathbb{P}_n(\mathbb{C})$)
(1) Let $\mathbb{P}_d(\mathbb{C}) \subset \mathbb{P}_n(\mathbb{C})$ be the inclusion

$$\langle z_0, \ldots, z_d \rangle \mapsto \langle z_0, \ldots, z_d, 0, \ldots, 0 \rangle.$$

Verify that $P_d(\mathbb{C})$ is a submanifold of dimension d.
(2) Let $Z \subset \mathbb{P}_n(\mathbb{C})$ be defined as the zero locus of the polynomial $f = z_0^2 + \cdots + z_d^2$, $0 < d \leqslant n$, and let

$$L = \{\langle x \rangle \in \mathbb{P}_n(\mathbb{C}) |\ x_0 = \cdots = x_d = 0\}.$$

(a) Show that L is the singular locus of Z.

(b) Let $Q = Z \cap \mathbb{P}_d(\mathbb{C})$ then Q is a submanifold of $\mathbb{P}_d(\mathbb{C})$ and of $\mathbb{P}_n(\mathbb{C})$ of dimension $d - 1$.

(c) Show that $L \cap Q = \emptyset$ and Z is the union of all those lines in $\mathbb{P}_n(\mathbb{C})$ connecting a point in L with a point in Q. (Z is then called the join of L and Q and denoted $L \vee Q$.)

Exercise 19: Let X and Y be complex manifolds with atlases $\mathcal{A}$ and $\mathcal{B}$ respectively. Let $X \times Y$ be the topological product.

(1) For $(U, \varphi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ show that $(U \times V, \varphi \times \psi)$ is a chart of $X \times Y$;

(2) show that the set of product charts $(U \times V, \varphi \times \psi)$, where $(U, \varphi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$, is a holomorphic atlas of $X \times Y$.

Exercise 20: Let $\Lambda \subset \mathbb{C}$ be a lattice.

(1) Let $\mathbb{C}/\Lambda \times \mathbb{C}/\Lambda \xrightarrow{\mu} \mathbb{C}/\Lambda$ be the operation of addition in the quotient group. Show that it is a holomorphic map.

(2) Let $a \in \mathbb{C}$ and let $\mathbb{C}/\Lambda \xrightarrow{\alpha} \mathbb{C}/\Lambda$ be the map $[x] \mapsto [a+x]$. Show that α is a biholomorphic map.

Exercise 21: (1) Let $\Lambda, \Lambda' \subset \mathbb{C}$ be lattices and let $\alpha \in \mathbb{C} \setminus \{0\}$ such that $\alpha\Lambda \subset \Lambda'$. Show that the multiplication map

$$\mathbb{C}/\Lambda \xrightarrow{\cdot\alpha} \mathbb{C}/\Lambda'$$

defined by $[z] \mapsto [\alpha z]$ is well-defined and holomorphic.

(2) Show that the map $\cdot\alpha$ is biholomorphic if and only if $\alpha\Lambda = \Lambda'$.

Exercise 22: Show that a meromorphic function on a torus $\mathbb{C}/\Lambda$ has equal numbers of zeroes and poles, both counted with multiplicities.

Hint: consider a meromorphic function f in $\mathcal{M}(\mathbb{C})$ and use the integral $\int\limits_{\partial V_a} f'(z)/f(z)\, dz$ for a suitable choice of a.

Exercise 23: Let f be a meromorphic function on $\mathbb{C}/\Lambda$, let $[a_1], \ldots, [a_m]$ be its zeroes and $[b_1], \ldots, [b_m]$ be its poles. Assume that each $[a_\mu]$ or $[b_\nu]$ is a simple zero or pole and they all are different. Show that

$$[a_1] + \cdots + [a_m] = [b_1] + \cdots + [b_m]$$

as elements of the group $\mathbb{C}/\Lambda$.

Hint: see previous exercise and consider an integral of $zf'(z)/f(z)$.

Exercise 24: Let $\gamma = p\tau + q$ be a point of the lattice $\mathbb{Z}\tau \oplus \mathbb{Z}$. Let $\xi = a\tau + b$, $a, b \in \mathbb{R}$, and let

$$\begin{aligned} e_\xi(\gamma, z) = &\exp(2\pi i a\gamma - \pi i p^2\tau - 2\pi i p(z+\xi)) = \\ &\exp(2\pi i a\gamma - \pi i p^2\tau - 2\pi i p\xi)\exp(-2\pi i p z) \end{aligned}$$

be the factor of automorphy of the theta function θ_ξ. Verify the rule

$$e_\xi(\gamma_1 + \gamma_2, z) = e_\xi(\gamma_1, \gamma_2 + z)e_\xi(\gamma_2, z)$$

for $\gamma_1, \gamma_2 \in \Gamma$.

Exercise 25: Let $(f_n)_{n\in\mathbb{Z}}$ be the family of meromorphic functions on $\mathbb{C}$ defined as $f_0(z) = \frac{1}{z}$ and $f_n(z) = \frac{1}{z-n} + \frac{1}{n}$ for $n \neq 0$.

(a) Show that $(f_n)_{n\in\mathbb{Z}}$ is locally uniformly summable over $\mathbb{C}$ (choose the exhaustion of $\mathbb{C}$ with $G_n = \{z = x + iy \in \mathbb{C} | \ -n < x, y < n\}$).

(b) Show that the resulting meromorphic function

$$f(z) = \frac{1}{z} + \sum_{n\neq 0} \left(\frac{1}{z-n} + \frac{1}{n} \right)$$

has as principal parts exactly the functions $\frac{1}{z-n}$, $n \in \mathbb{Z}$.

(c) Show that f can be written as

$$f(z) = \frac{1}{z} + \sum_{n\geqslant 1} \left(\frac{1}{z-n} + \frac{1}{z+n} \right),$$

with locally uniformly summable family $(g_n)_{n\geqslant 1}$, $g_n(z) = \frac{1}{z-n} + \frac{1}{z+n}$.

(d) Show that $f(z) = \pi \cot(\pi z)$.
(*Hint:* let $g(z) = f(z) - \pi \cot(\pi z)$; prove that $g'(z) + g'(z + 1/2) = 4g'(2z)$ and verify that $g' = 0$ and $g = 0$.)

(e) Show that the family $(\frac{1}{z-n})_{n\in\mathbb{Z}}$ is not locally uniformly summable.

Exercise 26: (a) Let $\mathbb{C}/\mathbb{Z}$ be the quotient group of $(\mathbb{C}, +)$ modulo the subgroup $(\mathbb{Z}, +)$, let $\mathbb{C}/\mathbb{Z}$ be endowed with the quotient topology. Show in analogy to tori $\mathbb{C}/\Lambda$ that $\mathbb{C}/\mathbb{Z}$ is a Riemann surface with natural charts defined by $\mathbb{C} \xrightarrow{\pi} \mathbb{C}/\mathbb{Z}$. ($\mathbb{C}/\mathbb{Z}$ is a cylinder).

(b) The functions $\sin(2\pi z)$, $\cos(2\pi z)$, $\exp(2\pi z)$ define holomorphic functions on $\mathbb{C}/\mathbb{Z}$, and the function $\cot(\pi z) = \frac{\cos(\pi z)}{\sin(\pi z)}$ defines a meromorphic function on $\mathbb{C}/\mathbb{Z}$.

Let $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and let $\wp$ be the Weierstraß function with respect to this lattice. Let

$$e_1 = \wp\left(\frac{\omega_1}{2}\right), \quad e_2 = \wp\left(\frac{\omega_2}{2}\right), \quad e_3 = \wp\left(\frac{\omega_1 + \omega_2}{2}\right).$$

Exercise 27: (1) The set of zeroes of $\wp'$ on $\mathbb{C}/\Lambda$ consists of $[\frac{\omega_1}{2}]$, $[\frac{\omega_2}{2}]$, and $[\frac{\omega_1}{2} + \frac{\omega_2}{2}]$, and these are simple zeroes (of order 1).

Exercise 28: (1) Show that $\wp'(z)^2 = (\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$.

(2) Show that $g_2 = e_1e_2 + e_1e_3 + e_2e_3$, $g_3 = -e_1e_2e_3$, $e_1 + e_2 + e_3 = 0$.

(3) Show that $g_2^3 - 27g_3^2 = 16(e_1 - e_2)^2(e_1 - e_3)^2(e_2 - e_3)^2$.

Hint: one could use the notion of a discriminant of the polynomial $4x^3 - g_2x - g_3$, if known from algebra.

(4) Show that $\wp''(z) = 6\wp(z)^2 - g_2/2$, $\wp''(\frac{\omega_1}{2}) = 2(e_1 - e_2)(e_2 - e_3)$.

Exercise 29: Let $\mathbb{C}/\Lambda \xrightarrow{\wp} \hat{\mathbb{C}}$ be the mapping of the meromorphic function $\wp$. Show that the fibres of $\wp$ over points different from e_1, e_2, e_3, ∞ consist of exactly 2 points, whereas these values are attained at $[\frac{\omega_1}{2}]$, $[\frac{\omega_2}{2}]$, and $[\frac{\omega_1}{2} + \frac{\omega_2}{2}]$, $[0]$ with multiplicity 2.