FB Mathematik
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WS 2008/2009

Exercises Algebraic Geometry I

(1) Let $f = x_n^s + a_1 x_n^{s-1} + a_2 x_n^{s-2} + \ldots + a_s$, with $a_i \in K[x_1, \ldots, x_{n-1}]$. Suppose that f is square–free (has no quadratic factors). Prove that the discriminant of f is nonzero.

(2) (a) Consider the curve defined by $y^2 - xy + x^2 = 0$, and the projection π on the x–axis. Compute for which points on the $x - axis$ the inverse image of π consists of less than two points.

(b) Consider the cone $x^2 - yz$, and the projection π on the y, z–plane. Compute for which points on the y, z–plane the inverse image of π consists of less than two points. What about projection on the x, y–plane?

(3) Let $f \in \mathbb{C}[x_1, \ldots, x_n]$. Show that if $f \neq 0$, then there exists $a \in \mathbb{C}^n$ with $f(a) \neq 0$.

(4) It is not true in general that $\mathcal{I}(V \cap W) = \mathcal{I}(V) + \mathcal{I}(W)$. Give a counterexample.

(5) (a) Show that $K[V]$ is isomorphic to $\mathbb{C}[x_1, \ldots, x_n]/\mathcal{I}(V)$.

(b) For a subset $F \subset K[V]$, define

$$V(F) := \{a \in V : f(a) = 0 \text{ for all } f \in F\}.$$

$V(F)$ is called the zero set of F. Show that the Nullstellensatz holds in the ring $K[V]$.

(c) Let $a, b \in V$. Show that there exists a regular function $f \in K[V]$ with $f(a) \neq f(b)$. (*Hint*: reduce to the case $V = \mathbb{C}^n$.)

(6) Let I, J be ideals in the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$. Suppose I is radical. Prove that $V(I : J) = \overline{V(I) \setminus V(J)}$.

(7) (a) Let $I \subset \mathbb{C}[x_1, \ldots, x_n]$ be an ideal, such that the vector space $\mathbb{C}[x_1, \ldots, x_n]/I$ has finite dimension s. Show that the set $V(I)$ consists of **at most** s points.

(b) Suppose, moreover, that I is radical. Show that $V(I)$ consists of exactly s points.

(c) Show that for a Noether normalisation $\pi : X \longrightarrow \mathbb{C}^k$, with s the degree of the corresponding field extension

$$\mathbb{C}(x_1, \ldots, x_k) \subset K(X) = Q(\mathbb{C}[X]),$$

the number of points in a **fibre** is **at most** s.

(8) Consider the map $f : \mathbb{C} \to \mathbb{C}^3$ defined by $f(t) = (t^4, t^5, t^6)$. Compute the ideal $I\left(\overline{f(\mathbb{C})}\right)$.

(9) Let $X = V(f_1, \ldots, f_s) \subset \mathbb{C}^n$, $Y = V(g_1, \ldots, g_t) \subset \mathbb{C}^m$, $f_1, \ldots, f_s \in \mathbb{C}[x_1, \ldots, x_n]$, $g_1, \ldots, g_t \in \mathbb{C}[y_1, \ldots, y_m]$. Show that the Cartesian product $X \times Y$ is also an affine variety, as zero–set of the ideal

$$\langle f_1, \ldots, f_s,\ g_1, \ldots, g_t \rangle \text{ in } \mathbb{C}[x_1, \ldots, x_n,\ y_1, \ldots, y_m].$$

(10) Let $V \subset \mathbb{C}^n$ be an algebraic variety. Prove that $I(V) = \bigcap_{p \in V} M_p$, here $M_p = \langle x_1 - p_1, \ldots, x_n - p_n \rangle$ for $p = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$.

(11) Let $\varphi : V \to W$ be a regular map such that φ^* is injective. Assume that $K[V]$ is a finitely generated $K[W]$–module (via φ^*). Prove that φ is surjective and the fibres consist of finitely many points.

(12) Determine the irreducible components of the following algebraic varieties.

- $V(x^2 - y^4, y^4 - x^2y^2 + xy^2 - x^3) \subset \mathbb{C}^2$;
- $V(x^2 + y^2 - 1, x^2 - z^2 - 1) \subset \mathbb{C}^3$.

(13) Determine the ideals of the irreducible components of $V = V(I) \subset \mathbb{C}^3$ for the following 3 cases of $I \subset \mathbb{C}[x, y, z]$.

- $I = (x^2 - yz, x - xz)$;
- $I = (x - yz, x^3 - z^3)$;
- $I = (x^2 + y^2 + z^2, x^2 - y^2 - z^2 + 1)$.

(14) Let $I \subset \mathbb{C}[x, y, z]$ be generated by $x^2 - y^3$ and $y^2 - z^3$.

a) Show that I is a prime ideal.

b) Find a surjective morphism $\mathbb{C} \to V(I)$ and check whether this is an isomorphism.

(15) Let X be a topological space, let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of abelian groups on X and let $f : \mathcal{F} \to \mathcal{G}$ be a homomorphism of sheaves.

- Prove that $f|_U$ is an injective map for all open subsets $U \subset X$ if and only if f_x is injective for all $x \in X$.
- Prove that $f|_U$ is an isomorphism for all open subsets $U \subset X$ if and only if f_x is an isomorphism for all $x \in X$.

(16) Give an example that the corresponding statement of (1) for "surjective" is wrong.

(17) Compute $\mathcal{O}_{\mathbb{C}^n}(\mathbb{C}^n \setminus \{0\})$.

(18) Is $V(x^3 - y^2) \subset \mathbb{C}^2$ isomorphic to $\mathbb{C}$?

(19) Let $U \subset \mathbb{C}^n$ be a non-empty open set. Show that the restriction map

$$\mathcal{O}_{\mathbb{C}^n}(\mathbb{C}^n) \to \mathcal{O}_{\mathbb{C}^n}(U)$$

is injective.

(20) Let $U = \mathbb{C}^n \setminus V(g_1, \ldots, g_m)$ for polynomials $g_1, \ldots, g_m \in \mathbb{C}[x_1, \ldots, x_n]$ such that

$$\gcd(g_1, \ldots, g_m) = 1.$$

Show that the restriction map

$$\mathcal{O}_{\mathbb{C}^n}(\mathbb{C}^n) \to \mathcal{O}_{\mathbb{C}^n}(U)$$

is bijective.

(21) Let X, Y be pre-varieties. Prove that $X \times Y$ is a pre-variety.

(22) Let $f : X \to Y$ be a morphism of pre-varieties. Prove that f is an isomorphism if and only if f is a homeomorphism and

$$f_x^* : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$$

is an isomorphism for all $x \in X$.

(23) Let X and Y be irreducible pre-varieties and let $f : X \to Y$ be a morphism. Prove that $f(X)$ is dense in Y if and only if

$$f_x^* : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$$

is injective for all $x \in X$.

(24) Let the map $\mathbb{P}^1(\mathbb{C}) \xrightarrow{f} \mathbb{P}^n(\mathbb{C})$ be given by

$$(t_0 : t_1) \mapsto (t_0^n : t_0^{n-1}t_1 : \ldots : t_1^n).$$

Show that f is injective and that $C_n = f(\mathbb{P}^1(\mathbb{C}))$ is a projective algebraic variety in $\mathbb{P}^n(\mathbb{C})$ defined by the condition

$$\operatorname{rk} \begin{pmatrix} x_0 & x_1 & \ldots & x_{n-1} \\ x_1 & x_2 & \ldots & x_n \end{pmatrix} \leqslant 1.$$

(25) Let $I \subset \mathbb{C}[x_1, \ldots, x_n]$ be an ideal and let $I^h \subset \mathbb{C}[x_0, \ldots, x_n]$ be the homogenization. Prove that $PV(I^h)$ is the closure of $V(I) \subset \mathbb{C}^n$ embedded in $\mathbb{P}^n$ by $(x_1, \ldots, x_n) \mapsto (1 : x_1 : \ldots : x_n)$.

(26) Let $X = PV(I) \subseteq \mathbb{P}^n$ be a projective variety such that

$$X_i = V\big(I(x_i = 1)\big) \subset \mathbb{C}^n$$

is not empty. Prove that X is irreducible if and only if X_i is irreducible and $X = \overline{X}_i$. Here X_i is embedded in $\mathbb{P}^n$ by

$$(x_1, \ldots, x_n) \mapsto (x_0 : \ldots : x_{i-1} : 1 : x_{i+1} : \ldots : x_n).$$

(27) Let $n \geqslant 2$ and $p \in \mathbb{P}^n$. Prove that $\mathbb{P}^n \setminus \{p\}$ is not isomorphic to an affine variety.

(28) Let $X \subset \mathbb{P}^3$ be the twisted cubic curve defined by the parametrization

$$\mathbb{P}^1 \to \mathbb{P}^3, \quad (s : t) \mapsto (s^3 : s^2t : st^2 : t^3).$$

(Note that this is a special case of Exercise (1) from Exercise sheet 8). Let $p = (0 : 0 : 1 : 0) \in \mathbb{P}^3$ and $H = PV(x_2)$. Let $\varphi : X \to H$ be defined by the rule $\varphi(q) = z$, where z is the intersection point of the line through q and p with H.

(a) Prove that φ defines a morphism of algebraic varieties.

(b) Prove that $\varphi : X \to \varphi(X)$ is not an isomorphism.

(29) (*Veronese surface*)

Let $\mathbb{P}^2 \xrightarrow{g} \mathbb{P}^5$ be the map defined by

$$(x_0 : x_1 : x_2) \mapsto (x_0^2 : x_0x_1 : x_0x_2 : x_1^2 : x_1x_2 : x_2^2).$$

(1) Verify that g is injective.

(2) Verify that the image $V = g(\mathbb{P}^2)$ is a projective algebraic variety and find generators of its homogeneous ideal.

(3) Let $S = \mathbb{C}[z_0, z_1, z_2]$ and let $S_2 \subset S$ be the subvectorspace over $\mathbb{C}$ of polynomials of degree 2. Then using the basis

$$\{z_0^2, z_1^2, z_2^2, 2z_0z_1, 2z_0z_2, 2z_1z_2\}$$

of S_2 one can consider $\mathbb{P}(S_2) \simeq \mathbb{P}^5$ as the space of all conics $V(f) \subset \mathbb{P}^2$, $f \in S_2$, $f \neq 0$. Show that under this identification the Veronese surface $V \subset \mathbb{P}^5$ consists of those conics $V(f)$ with $f = a^2$, where a is a linear form. Equivalently: V consists of those conics which are double lines.

(30) (*central projection*)

Let $p = (0 : \cdots : 0 : 1) \in \mathbb{P}^{n+1}$.

(a) Show that the map

$$\mathbb{P}^{n+1} \setminus \{p\} \xrightarrow{\pi} \mathbb{P}^n, \quad (x_0 : \cdots : x_{n+1}) \mapsto (x_0 : \cdots : x_n)$$

is well-defined and is a surjective morphism.

(b) Show that the fibres of π are isomorphic to k as algebraic varieties.

(c) Show that for any standard open affine set (chart) $U_i \subset \mathbb{P}^n$ there is an "obvious" isomorphism (of algebraic varieties)

$$\pi^{-1}(U_i) \xrightarrow{\tau_i} U_i \times k,$$

such that $p_i \circ \tau_i = \pi|_{\pi^{-1}(U_i)}$, where p_i is the first projection $U_i \times k \to U_i$.

(d) Calculate the map $\tau_i \circ \tau_j^{-1} : U_{ij} \times k \to U_{ij} \times k$, where $U_{ij} = U_i \cap U_j$.

(e) Let $\mathbb{P}^n \subset \mathbb{P}^{n+1}$ be embedded via $(x_0 : \cdots : x_n) \mapsto (x_0 : \cdots : x_n : 0)$. Show that the fibres of π are the lines connecting the points of $\mathbb{P}^n$ with p (with p deleted). Draw a picture of this situation.

(31) Prove that every irreducible conic in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$.

(32) Prove that the quadratic Cremona transformation

$$\varphi : \mathbb{P}^2 \to \mathbb{P}^2, \quad \varphi(x_0 : x_1 : x_2) = (x_1x_2 : x_0x_2 : x_0x_1)$$

is birational. Find $\mathrm{Dom}(\varphi)$ and $\mathrm{Dom}(\varphi^{-1})$.

(33) Prove that the hypersurface $V_+(x_0x_1 - x_2x_3) \subset \mathbb{P}^3$ is birationally equivalent to $\mathbb{P}^2$ but is not isomorphic to $\mathbb{P}^2$.

(34) Let $X \subset \mathbb{P}^2$, $X \neq \mathbb{P}^2$, be an irreducible projective subvariety in $\mathbb{P}^2$ having at least two different points. Show that $X = V_+(f)$ for some homogeneous irreducible polynomial f.

Hint. One should understand that non-trivial (different from zero and from the whole ring) prime ideals in $\mathbb{C}[x, y]$ are either maximal ideals or those generated by a single irreducible polynomial.

(35) Prove that $\mathbb{P}^1 \times \mathbb{P}^1$ is not isomorphic to $\mathbb{P}^2$.

Hint. Consider $C_1 = \{(1:0)\} \times \mathbb{P}^1$ and $C_2 = \{(1:0)\} \times \mathbb{P}^1$. They are closed subvarieties in $\mathbb{P}^1 \times \mathbb{P}^1$ that have no common points. Assume that $\mathbb{P}^1 \times \mathbb{P}^1 \simeq \mathbb{P}^2$ and consider the images of C_1 and C_2 under this isomorphism. Prove that they must intersect. This contradiction will show that $\mathbb{P}^1 \times \mathbb{P}^1 \not\simeq \mathbb{P}^2$.

(36) Let $Y = V(y^2 - x^3) \subset \mathbb{C}^2$. Let $\widetilde{Y}$ be the strict transform of Y under the blowing up of $\mathbb{C}^2$ at $0 = (0, 0)$. Prove that the projection $\widetilde{Y} \longrightarrow Y$ is a homeomorphism but not an isomorphism.

(37) Let $f : X \to Y$ be a finite morphism of varieties. Show that the induced morphism

$$X \to \overline{f(X)}$$

is a finite morphism as well.

(38) (a) Let $f_1 : Y_1 \to Y_2$, $f_2 : X_1 \to X_2$ be finite morphisms. Prove that the morphism $f_1 \times f_2 : Y_1 \times X_1 \to Y_2 \times X_2$ is finite.

(b) Let $f : X \to Y$ and $g : Y \to Z$ be finite, prove that $g \circ f$ is finite.

(39) Let X and Y be algebraic varieties, let Y be complete, and let $f : X \to Y$ be a finite morphism. Prove that X is complete.

(40) (*Hypersurfaces in* $\mathbb{P}^n$). Let $f \in \mathbb{C}[x_0, \ldots, x_n]$ be a homogeneous polynomial of degree $d \geqslant 1$. Consider $V_+(f) \subset \mathbb{P}^n$.

(a) Prove (or understand) the folowing formula (formula of Euler)

$$f = \frac{1}{d} \sum_{\nu=0}^{n} x_\nu \frac{\partial f}{\partial x_\nu}.$$

(b) Show that $V_+(f)$ is a smooth algebraic variety if and only if

$$\left(\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n} \right) (p) \neq 0 \quad \text{for all } p \in V_+(f).$$

(41) Let C be a conic in $\mathbb{P}^2$, i. e., a zero set $V_+(f)$ of a homogeneous polynomial $f \in \mathbb{C}[x_0, x_1, x_2]$ of degree 2. Prove that C is smooth if and only if C is irreducible.

(42) Let $f = x_1^3 + x_2^3 + x_3^3 + x_4^3$ and $g = x_3^2 - x_1x_2$ be two polynomials in $\mathbb{C}[x_1, x_2, x_3, x_4]$.

(a) Determine the singular locus of the affine algebraic varieties $V(f)$ and $V(g)$.

(b) Determine the singular locus of $V(f, g)$.