University of Luxembourg
Mathematics Research Unit
Prof. Dr. Martin Schlichenmaier
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SS 2013
Exercises from Lecture 1

Infinite dimensional Lie algebras

Exercise 1: Let L be a Lie algebra. Let $x \in L$. Then the centralizer of x is by definition the set

$$C(x) = \{y \in L \mid [x, y] = 0\}.$$

Show that for every $x \in L$ it's centralizer $C(x)$ is a Lie subalgebra of L.

Exercise 2: In the lecture we defined the Witt Lie algebra $W = \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[z, z^{-1}])$. We also found a basis $\{L_n\}_{n \in \mathbb{Z}}$ of W defined by

$$L_n = z^{n+1} \frac{\partial}{\partial z}.$$

Using this basis, for a given $D \in W$ compute the centralizer $C(D)$.

Exercise 3: Describe all abelian subalgebras of W.

Exercise 4: Describe the Lie algebra $\mathrm{Der}_{\mathbb{K}}(A)$ and its abelian subalgebras, for $A = \mathbb{K}[t]$ and $A = \mathbb{K}[x, y]/(xy - 1)$.

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Exercises from Lecture 2

Infinite dimensional Lie algebras

Exercise 5: Consider the chain of $\mathbb{K}$-subalgebras $\mathbb{K}[z] \subset \Bbbk[z, z^{-1}] \subset \mathbb{K}(z)$.

1) Show that every derivation of $\mathbb{K}[z]$ extends uniquely to a derivation of $\Bbbk[z, z^{-1}]$ and every derivation of $\Bbbk[z, z^{-1}]$ extends uniquely to a derivation of $\mathbb{K}(z)$. Conclude that there is the chain of Lie subalgebras

$$\mathrm{Der}(\mathbb{K}[z]) \subset \mathrm{Der}(\Bbbk[z, z^{-1}]) \subset \mathrm{Der}(\mathbb{K}(z)).$$

2) Let $\mathbb{K} = \mathbb{C}$. Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere (projective line). Then $\mathrm{Der}(\mathbb{C}(z))$ can be realized as the Lie algebra of meromorphic vector fields on $\hat{\mathbb{C}}$. In the lecture we interpreted the Witt algebra $W = \mathrm{Der}(\mathbb{C}[z, z^{-1}])$ as the subalgebra of meromorphic vector fields on $\hat{\mathbb{C}}$ that are holomorphic on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Give a similar description of the Lie algebra $\mathrm{Der}(\mathbb{C}[z])$.

Exercise 6: In the lecture we showed that $\langle L_{-1}, L_0, L_1 \rangle_{\mathbb{C}}$ is a Lie subalgebra of the Witt algebra W isomorphic to $sl(2, \mathbb{C})$. It can be understood as the subalgebra of holomorphic vector fields on $\hat{\mathbb{C}}$.

1) For every natural number n, show that $\langle L_{-n}, L_0, L_n \rangle_{\mathbb{C}}$ is a Lie subalgebra of W isomorphic to $sl(2, \mathbb{C})$.

2) Give a geometrical description of this subalgebra.

Exercise 7: Describe all finite dimensional Lie subalgebras of the Witt algebra W.

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Exercises from Lecture 3

Infinite dimensional Lie algebras

Exercise 8: Describe all representations of $sl(2,\mathbb{C})$ of dimension $\leqslant 3$.

Exercise 9: Let $\mathbb{K}$ be a field. In the lecture we defined the Heisenberg algebra $\mathcal{H}$ over $\mathbb{K}$ as a 3-dimensional Lie algebra over $\mathbb{K}$ with generators p, q, I and relations

$$[p,q] = I, \quad [p,I] = 0, \quad [q,I] = 0.$$

1) Can you find a non-trivial finite dimensional representation of $\mathcal{H}$? Is there any difference between the cases $\operatorname{char}\mathbb{K} = 0$ and $\operatorname{char}\mathbb{K} \neq 0$?

2) Let $\operatorname{char}\mathbb{K} = 2$. Construct a non-trivial 2-dimensional representation of $\mathcal{H}$.

Exercise 10: Describe all finite dimensional representations of the Witt algebra W.

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Exercises from Lectures 4 & 5

Infinite dimensional Lie algebras

Exercise 11: Let $\widehat{gl(\infty)}$ be the set of matrices $(a_{ij})_{i,j\in\mathbb{Z}}$ with entries from a base field $\Bbbk$ such that in every row and in every column there are only finitely many non-zero entries.

1) Show that one can define multiplication on $\widehat{gl(\infty)}$ by the standard formula.

2) Show that this makes $\widehat{gl(\infty)}$ an associative $\Bbbk$-algebra and hence a Lie algebra.

3) Conclude that $\widehat{gl(\infty)}$ contains as a subalgebra (Lie and associative one) the algebra $\overline{gl(\infty)}$ of all matrices $(a_{ij})_{i,j\in\mathbb{Z}}$ with finitely many diagonals.

4) Check whether $\overline{gl(\infty)}$ is an ideal in $\widehat{gl(\infty)}$.

5) What can you say about the dimension of the quotient vector space $\widehat{gl(\infty)}/\overline{gl(\infty)}$.

Exercise 12: Let L be a Lie algebra and let M be an L-module. Give an explicit description of $Z^0(L,M)$, $B^0(L,M)$, $H^0(L,M)$, $Z^1(L,M)$, $B^1(L,M)$, and $H^1(L,M)$.

Exercise 13: Consider the natural 2-dimensional representation of $sl(2,\Bbbk)$, i. e., $L = sl(2,\Bbbk)$, $M = \Bbbk^2$, and $a \cdot v$ is the usual multiplication for $a \in sl(2,\Bbbk)$ and $v \in \Bbbk^2$. Compute $H^i(L,M)$ for $i = 0,1$.

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Exercises from Lecture 6

Infinite dimensional Lie algebras

Exercise 14: Recall that the algebra $\mathcal{A} = \Bbbk[z, z^{-1}]$ of Laurent polynomials in one variable is a module over the Witt algebra $W = \mathrm{Der}(\mathcal{A})$. Consider the vector space $D = \mathcal{A} \oplus W$ and define

$$[(a_1, \delta_1), (a_2, \delta_2)] := (\delta_1 \cdot a_2 - \delta_2 \cdot a_1, [\delta_1, \delta_2]).$$

Prove that this introduces a structure of a Lie algebra on D.

Exercise 15: In the notations of the previous exercise, show that there is a short exact sequence of Lie algebras

$$0 \to \mathcal{A} \xrightarrow{i} D \xrightarrow{p} W \to 0$$

with $i(a) = (a, 0)$, $p(a, \delta) = \delta$. Check that

$$s : W \to D, \quad \delta \mapsto (0, \delta),$$

is an injective homomorphism of Lie algebras such that $p \circ s = \mathrm{id}_W$. This makes it possible to consider W as a Lie subalgebra of D. Show that W is not an ideal of D.

Exercise 16: Let L be a non-abelian two-dimensional Lie algebra. Notice that L is unique up to an isomorphism. Let $\Bbbk$ be a trivial 1-dimensional L-module. Compute $H^2(L, \Bbbk)$ and describe all central extensions

$$0 \to \Bbbk \to \hat{L} \to L \to 0.$$

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Exercises from Lecture 7

Infinite dimensional Lie algebras

Exercise 17: Work out the missing details of the computation of $H^2(W, \Bbbk)$ for the Witt algebra W.

Exercise 18: (0) Let $P = \Bbbk[x, y]$. Let

$$[f, g] := \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}.$$

This gives P a structure of an infinite-dimensional Lie algebra.
Consider the adjoint representation

$$\mathrm{ad} : P \to \mathrm{Der}(\Bbbk[x, y]), \quad f \mapsto \mathrm{ad}(f), \quad \mathrm{ad}(f)(g) = [f, g].$$

Prove that its image coincides with the subalgebra H of $\mathrm{Der}(\Bbbk[x, y])$ consisting of the derivations $D = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$ with zero divergence

$$\mathrm{div} D := \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 0.$$

Notice that its kernel coincides with the center $Z(P)$ of P and equals $\Bbbk \subset P$. Hence there is a central extension

$$0 \to \Bbbk \to P \to H \to 0.$$

(1) Prove that P is a non-trivial central extension of H. Write down the corresponding 2-cocycle $\alpha \in Z^2(H, \Bbbk)$.

(2) Consider the trivial extension $\Bbbk \oplus H$ and describe it as a subalgebra of $gl(\Bbbk[x, y]) = \mathrm{End}_{\Bbbk}(\Bbbk[x, y])$.

(3) Mimic the computations we did in the lecture for the Witt algebra and compute $H^2(H, \Bbbk)$. Describe all central extensions of H.

Exercise 19: (1) Let $\overline{gl(\infty)}$ be the Lie algebra of all infinite matrices with finitely many diagonals. Let $A = (a_{ij}) \in \overline{gl(\infty)}$. Let $\pi(A)$ be the matrix with entries

$$(\pi(A))_{ij} = \begin{cases} a_{ij}, & i, j \geqslant 0; \\ 0, & \text{otherwise.} \end{cases}$$

Let $\rho(A, B) = \pi([A, B]) - [\pi A, \pi B]$. Prove that it makes sense to speak about the trace of $\rho(A, B)$ and define

$$\alpha(A, B) := \mathrm{Tr}\, \rho(A, B).$$

(2) Prove that α is a non-trivial 2-cocycle, i. e., $\alpha \in Z^2(\overline{gl(\infty)}, \Bbbk)$ and $[\alpha] \neq 0$ as element of $H^2(\overline{gl(\infty)}, \Bbbk)$.

(3) Let E_{ij} denote the matrix such that its only non-zero entry at ij equals 1. Prove that

$$\alpha(E_{ij}, E_{ml}) = \begin{cases} 1, & i = l \geqslant 0, \quad j = m < 0; \\ -1, & j = m \geqslant 0, \quad i = l < 0; \\ 0, & \text{otherwise.} \end{cases}$$

Let $\hat{L} = \Bbbk \oplus_{\Bbbk} \overline{gl(\infty)}$ be the central extension corresponding to α. Let $t = (1, 0)$ and let $\hat{E}_{ij} = (0, E_{ij})$. Show that the Lie bracket is given by

$$[\hat{E}_{ij}, \hat{E}_{ml}] = \delta_{jm} \cdot \hat{E}_{il} - \delta_{il}\hat{E}_{mj} + \alpha(E_{ij}, E_{ml}) \cdot t$$

(4) Consider the restriction of α to the subalgebra $gl(\infty) \subset \overline{gl(\infty)}$. Does $\alpha|_{gl(\infty)}$ represent a non-trivial element from $H^2(gl(\infty), \Bbbk)$?