FB Mathematik
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Exercise class for the course "Algebraic Geometry".

Exercise 1: Show that $k[x_1, \ldots, x_n] \cong k[x_1, \ldots, x_{n-1}][x_n]$.

Exercise 2: Let $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$, $a_\nu \in k$. Show that $k[x_1, \ldots, x_n]/\mathfrak{m} \cong k$.

Exercise 3: Show that $(x_1, \ldots, x_p) \subset k[x_1, \ldots, x_n]$, $p \leqslant n$, is a prime ideal.

Exercise 4: Show that $x^3 - y^2 \in k[x, y]$ is irreducible.

Exercise 5: Prepare a talk on §1 of the script "Garben".

Exercise 6: Determine the irreducible components of the algebraic set $A \subset \mathbb{C}^2$ given by the ideal

$$(x^2 - y^4, y^4 - x^2y^2 + xy^2 - x^3) \subset \mathbb{C}[x, y].$$

Exercise 7: Determine the irreducible components of the algebraic set $A \subset \mathbb{C}^3$ given by the ideal

$$(x^2 + y^2 - 1, x^2 - z^2 - 1) \subset \mathbb{C}[x, y, z].$$

Exercise 8: Let $A := \{(t, t^2, t^3) \in \mathbb{C}^3 \mid t \in \mathbb{C}\}$.

a) Show that A is an algebraic set and determine its ideal.

b) Show that A is irreducible.

In exercises 9, 10, and 11 the ground field k is assumed to be algebraically closed.

Exercise 9: Determine the ideals of the irreducible components of $Z = Z(\mathfrak{a}) \subset k^3$ for the following 3 cases of $\mathfrak{a} \subset k[x, y, z]$.

- $\mathfrak{a} = (x^2 - yz, x - xz)$
- $\mathfrak{a} = (x - yz, x^3 - z^3)$
- $\mathfrak{a} = (x^2 + y^2 + z^2, x^2 - y^2 - z^2 + 1)$

Exercise 10: Let $Z = \{p_1, \dots, p_d\} \subset k^n$ be a set of d points. Show that

a) Z is an algebraic set with $i(Z) = \mathfrak{m}(p_1) \cap \dots \cap \mathfrak{m}(p_d)$;

b) the coordinate ring $A(Z)$ is isomorphic to the product of rings

$$R_n/\mathfrak{m}(p_1) \times \dots \times R_n/\mathfrak{m}(p_d),$$

where $R_n = k[x_1, \dots, x_n]$.

Exercise 11: a) Let $\mathfrak{a} \subset k[x_1, \dots, x_m]$ and $\mathfrak{b} \subset k[y_1, \dots, y_n]$ be ideals. Show that the set

$$Z(\mathfrak{a}) \times Z(\mathfrak{b}) \subset k^m \times k^n = k^{m+n}$$

is algebraic and equal to $Z(\tilde{\mathfrak{a}} + \tilde{\mathfrak{b}})$, where $\tilde{\mathfrak{a}} \subset k[x_1, \dots, x_m, y_1, \dots, y_n]$ is generated by $\mathfrak{a}$ and $\tilde{\mathfrak{b}} \subset k[x_1, \dots, x_m, y_1, \dots, y_n]$ is generated by $\mathfrak{b}$.

b) Show that $Z(\mathfrak{a}) \times Z(\mathfrak{b})$ is irreducible if each of the factors is irreducible.

c) Try to show that $\tilde{\mathfrak{a}} + \tilde{\mathfrak{b}}$ is reduced if $\mathfrak{a}$ and $\mathfrak{b}$ are reduced.

Hint to c). One can write any $\bar{f} \in k[x_1, \dots, x_m, y_1, \dots, y_n]/(\tilde{\mathfrak{a}} + \tilde{\mathfrak{b}})$ as

$$\bar{f} = \sum_{i=1}^{p} \bar{f}_i \bar{g}_i$$

with $f_i \in k[x_1, \dots, x_m]$, $g_i \in k[y_1, \dots, y_n]$. Let $f|_{Z(\mathfrak{a}) \times Z(\mathfrak{b})} = 0$ and assume that $\bar{f} \neq \bar{0}$ and $p > 0$ is minimal for that f. Show that there is an index i_0 and $b_0 \in Z(\mathfrak{b})$ with $g_{i_0}(b_0) \neq 0$. Conclude from this that the classes $\bar{f}_1, \dots, \bar{f}_p \in k[x_1, \dots, x_n]/\mathfrak{a}$ are linearly dependent over k and finally that p is not minimal.

Exercise 12: Let the ideal $\mathfrak{a} \subset k[x, y, z]$ be generated by $xz - y^2$, $yz - x^3$, $z^2 - x^2 y$.

a) Show that $Z(\mathfrak{a}) \subset k^3$ is irreducible.

b) Find a surjective morphism

$$(k, k[T]) \to (Z(\mathfrak{a}), k[x, y, z]/\mathfrak{a}).$$

Exercise 13: Let $\mathfrak{a} \subset k[x, y, z]$ be generated by $x^2 - y^3$ and $y^2 - z^3$.

a) Show that $\mathfrak{a}$ is a prime ideal.

b) Find a surjective morphism

$$(k, k[T]) \to (Z(\mathfrak{a}), k[x, y, z]/\mathfrak{a})$$

and check whether this is an isomorphism.

Exercise 14: Let (X, A) be an affine algebraic set over a field k, let $f \in A$, and let $X_f = X \setminus Z(f)$. Verify that (X_f, A_f) is again an affine algebraic set over k, where A_f is the ring of fractions $\frac{a}{f^n}$, $a \in A$.

Exercise 15: Draw a picture of $Z(x^2 - y^2 z)$ in k^3 using the intersections

$$Z(x^2 - y^2 z) \cap H_c,$$

where $H_c = Z(x - c)$, $c \in k$.

Exercise 16: Let $\mathfrak{a} \subset k[x_1, \dots, x_m] =: R_m$ and $\mathfrak{b} \subset k[y_1, \dots, y_n] =: R_n$ be ideals and let $\tilde{\mathfrak{a}}$ respectively $\tilde{\mathfrak{b}}$ be the ideals in $k[x_1, \dots, x_m, y_1, \dots, y_n] =: R_{m+n}$ generated by $\mathfrak{a}$ respectively $\mathfrak{b}$ in R_{m+n}. Verify that

$$R_m/\mathfrak{a} \otimes_k R_n/\mathfrak{b} \cong R_{m+n}/(\tilde{\mathfrak{a}} + \tilde{\mathfrak{b}}).$$

Hint. Prove first that $R_m \otimes_k R_n \cong R_{m+n}$ and then start with the diagram (to be explained)

$$\begin{array}{ccc} R_m \otimes_k R_n & \longrightarrow & R_m \otimes_k (R_n/\mathfrak{b}) \\ \downarrow & & \downarrow \\ (R_m/\mathfrak{a}) \otimes_k R_n & \longrightarrow & (R_m/\mathfrak{a}) \otimes_k (R_n/\mathfrak{b}) \end{array}$$

and complete that to an "exact 9-diagram".

Exercise 17: Let (X, A) be an affine algebraic set over $k = \bar{k}$ and let $\operatorname{Specm}(A)$ be the set of all maximal ideals of A.

a) Prove that for any maximal ideal $\mathfrak{m} \subset A$ the composition

$$\alpha_{\mathfrak{m}} : k \to A \to A/\mathfrak{m}$$

is an isomorphism.

b) Let for any $f \in A$ the map $\operatorname{Specm}(A) \to k$ be defined by $\mathfrak{m} \mapsto \alpha_{\mathfrak{m}}^{-1}(\bar{f}_{\mathfrak{m}})$, where $\bar{f}_{\mathfrak{m}}$ is the residue class of f in $A/\mathfrak{m}$. Show that (X, A) and $(\operatorname{Specm}(A), A)$ are isomorphic (as objects in the category (AffS/k)).

Exercise 18: Let $a(x) = a_0 x_0 + \dots + a_n x_n$ be a linear form and let

$$H = \{\langle x \rangle \in \mathbb{P}_n(k) \mid a(x) = 0\}$$

be the hyperplane defined by it. Prove that if $a_i \neq 0$, then the map

$$\langle x \rangle \mapsto \frac{1}{a(x)}(x_0, \dots, \hat{x}_i, \dots, x_n)$$

defines an "affine" chart $\mathbb{P}_n(k) \setminus H \xrightarrow{\simeq} k^n$.

Exercise 19: a) Let $H_1, \dots, H_m$ be hypersurfaces in $\mathbb{P}_n(k)$, $m \leqslant n$. Show that $H_1 \cap \dots \cap H_m \neq \emptyset$.

b) Let $L = \mathbb{P}U$ be a line ($\dim_k U = 2$) in $\mathbb{P}_n(k)$ and let $H \subset \mathbb{P}_n(k)$ be a hyperplane. Show that $L \cap H$ is a point if and only if $L \not\subset H$.

Exercise 20: Let the map $\mathbb{P}_1(k) \xrightarrow{f} \mathbb{P}_n(k)$ be given by $\langle t_0, t_1\rangle \mapsto \langle t_0^n, t_0^{n-1}t_1, \ldots, t_1^n\rangle$. Show that f is injective and that $C_n = f(\mathbb{P}_1(k))$ is a projective algebraic variety defined by the condition

$$\operatorname{rk}\begin{pmatrix} z_0 & z_1 & \ldots & z_{n-1} \\ z_1 & z_2 & \ldots & z_n \end{pmatrix} \leqslant 1.$$

Exercise 21: Let $f_0(z_0, z_1), f_1(z_0, z_1) \in k[z_1, z_2]$ be homogeneous polynomials of the same degree d without common zeros (except of $(0,0)$).

a) Show that the fibres of the map $\mathbb{P}_1(k) \to \mathbb{P}_1(k)$ defined by

$$\langle t_0, t_1\rangle \mapsto \langle f_0(t_0, t_1), f_1(t_0, t_1)\rangle$$

consists of at most d points and describe those points of $\mathbb{P}_1(k)$ over which there are less than d points by conditions on f_0 and f_1.

b) In case of $f_0(z_0, z_1) = z_0^2$, $f_1(z_0, z_1) = z_1^2$ show that there are affine charts $U \subset \mathbb{P}_1(k)$ which are not mapped into an affine chart.

Exercise 22: Let X, Y, S be affine algebraic sets with coordinate rings $A(X)$, $A(Y)$, and $A(S)$ respectively. Let $X \xrightarrow{f} S$, $Y \xrightarrow{g} S$ be morphisms. Let

$$X \times_S Y := \{(x, y) \in X \times Y \mid f(x) = g(y)\}$$

be the fibre product. Prove that

(1) $X \times_S Y$ is an affine algebraic set with coordinate ring

$$A(X \times_S Y) \simeq A(X) \otimes_{A(S)} A(Y);$$

(2) given two morphisms $Z \xrightarrow{u} X$ and $Z \xrightarrow{v} Y$ from an affine algebraic set Z with $f \circ u = g \circ v$, there is a unique morphism $Z \xrightarrow{w} X \times_S Y$ such that $u = p \circ w$, $v = q \circ w$, where p and q are the projections from $X \times_S Y$ to X and Y respectively.

Exercise 23: Let the Zariski topology (Z-topology) on $\mathbb{P}_n(k)$ be defined by declaring the algebraic subsets as closed. Show that the following statements hold true.

(1) The Z-topology on $\mathbb{P}_n(k)$ is the quotient topology of the (induced) Z-topology of $k^{n+1} \setminus \{0\}$ under the canonical surjection $k^{n+1} \setminus \{0\} \to \mathbb{P}_n(k)$.

(2) For each affine chart $\mathbb{P}_n(k) \setminus H \xrightarrow{\varphi} k^n$, where H is a hyperplane, φ is a homeomorphism.

(3) A subset $Q \subset \mathbb{P}_n(k)$ is open in the Z-topology if and only if $\varphi_i(Q \cap U_i)$ is open in k^n for $i = 0, \ldots, n$, where $U_i \xrightarrow{\varphi_i} k^n$ are the standard affine charts.

Exercise 24: Recall that linear transformations of $\mathbb{P}_n(k)$ are those given by the rule $\langle x\rangle \mapsto \langle A(x)\rangle$ with $A \in \mathrm{GL}_{n-1}(k)$.

(1) Let $p_1 = \langle 1, 0\rangle$, $p_2 = \langle 1, 1\rangle$, $p_3 = \langle 0, 1\rangle$ be points in $\mathbb{P}_1(k)$, and let q_1, q_2, q_3 be any three different from each other points in $\mathbb{P}_1(k)$. Show that there exists a unique linear transformation $\alpha : \mathbb{P}_1(k) \to \mathbb{P}_1(k)$ with $\alpha(p_i) = q_i$, $i = 1, 2, 3$.

(2) Let $x = \langle x_0, x_1\rangle$, $y = \langle y_0, y_1\rangle$, $z = \langle z_0, z_1\rangle$, $w = \langle w_0, w_1\rangle$ be pairwise different points in $\mathbb{P}_1$, let $(x, y) := x_0y_1 = x_1y_0$, $(x, z) := x_0z_1 - x_1z_0$ etc., and let

$$[x, y, z, w] := \frac{(y, x)(w, z)}{(w, x)(y, z)},$$

called the cross ratio of the four points x, y, z, and w. Show that the cross ratio is invariant under linear transformations.

Exercise 25: Let $U_1 = \mathbb{P}V_1$, $U_2 = \mathbb{P}V_2$ be projective linear subspaces of $\mathbb{P}V$, and let $U_1 + U_2$ be the smallest linear subspace of $\mathbb{P}V$ containing U_1 and U_2. Show that

$$\dim(U_1 \cap U_2) = \dim U_1 + \dim U_2 - \dim(U_1 + U_2).$$

Exercise 26: Let (X, A) be an irreducible algebraic set over k. For any domain R, denote by $Q(R)$ the field of fractions of R. Show that

(i) the natural maps $A \to A_g$ and $A \to A_{\mathfrak{m}(p)}$ are injective; the natural maps $A_g \to Q(A)$ and $A_{\mathfrak{m}(p)} \to Q(A)$ defined by $\frac{f}{g} \mapsto \frac{f}{g}$ (show that this is well-defined) are injective homomorphisms and one can identify $Q(A_g)$ and $Q(A_{\mathfrak{m}(p)})$ with $Q(A)$;

(ii) if $g(p) \neq 0$, then $A_g \subset A_{\mathfrak{m}(p)}$;

(iii) if $U \subset X$ is Zariski open, then also $\tilde{A}(U) \to A_{\mathfrak{m}(p)} \cong \tilde{A}_p$ is injective;

(iv) $A = \tilde{A}(X) \to \tilde{A}(U)$ is injective for all open non-empty sets $U \subset X$;

(v) $\tilde{A}(U) = \bigcap\limits_{p \in U} A_{\mathfrak{m}(p)}$ in $Q(A)$ and $Q(\tilde{A}(U)) = Q(A)$ (note that $\tilde{A}(U)$ is a domain);

(vi) if $U = k^n \setminus Z(g, h)$ with g, h without common factor, then the restriction map $A = \tilde{A}(k^n) \to \tilde{A}(U)$ is bijective (use $k^n(g), k^n(h) \subset U$, $\tilde{A}(U) \subset Q(A)$);

(vii) given U as in (vi), then U is not affine (assume $j : U \hookrightarrow k^N$ as an algebraic set, then use (vi) to obtain a morphism $k^n \to k^N$ extending j and leading to a contradiction);

(viii) conclude that an open set $U \subset k^n$ that is affine is necessarily a principal open subset $(k^n)(g) = k^n \setminus Z(g)$.

Exercise 27: Let $g \in S = k[z_0, \dots, z_n]$ be homogeneous and let $S_{(g)} \subset S_g$ be the subring

$$S_{(g)} := \left\{ \frac{f}{g^m} \mid f \text{ homogeneous of degree } m \deg(g) \right\}.$$

Show that $\mathcal{O}_{\mathbb{P}_n(k)}(D(g)) \simeq S_{(g)}$.

Exercise 28: Let $f_1, \ldots, f_m \in S$ be homogeneous of same degree with $Z(f_1, \ldots, f_m) = \{0\}$ in k^{n+1}. Verify that the map $\mathbb{P}_n(k) \xrightarrow{f} \mathbb{P}_m(k)$ defined by $f(\langle x \rangle) = \langle f_1(x), \ldots, f_m(x) \rangle$ is a morphism.

Exercise 29: (*Veronese surface*) Let $\mathbb{P}_2(k) \xrightarrow{g} \mathbb{P}_5(k)$ be the map defined by

$$\langle x_0, x_1, x_2 \rangle \mapsto \langle x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2 \rangle.$$

(1) Verify that g is injective.

(2) Verify that the image $V = g(\mathbb{P}_2(k))$ is a projective algebraic set and find generators of its homogeneous ideal.

(3) Let $S = k[z_0, z_1, z_2]$ and let $S_2 \subset S$ be the subvectorspace over k of polynomials of degree 2. Then $\mathbb{P}(S_2) \simeq \mathbb{P}_5(k)$ (using the basis $z_0^2, z_1^2, z_2^2, 2z_0z_1, 2z_0z_2, 2z_1z_2$ of S_2) can be considered as the space of all conics $Z(f) \subset \mathbb{P}_2$, $f \in S_2$, $f \neq 0$, via $Z(f) \leftrightarrow \langle f \rangle$.

Show that the Veronese surface $V \subset \mathbb{P}_5(k)$ consists of those $\langle f \rangle$ with $f = a^2$, where a is a linear form. Equivalently: V consists of those conics which are double lines.

Exercise 30: Let X be a prevariety over algebraically closed field $k = \bar{k}$ and let $Y \subset X$ be a closed subset. Let the sheaf $\mathcal{O}_Y \subset \mathcal{C}_Y$ be defined by

$$\mathcal{O}_Y(V) = \left\{ V \xrightarrow{f} k \;\middle|\; \begin{array}{c} \forall\, p \in V\ \exists\, U(p) \subset X \text{ open} \\ \exists F \in \mathcal{O}_X(U) \text{ such that } U \cap Y \subset V \\ f|_{U\cap Y} = F|_{U \cap Y} \end{array} \right\}$$

for an open subset $V \subset Y$ in the induced topology.

(a) Show that $(Y, \mathcal{O}_Y)$ is a prevariety over the field k.

(b) Let $\mathcal{I}_Y \subset \mathcal{O}_X$ be the sheaf of functions vanishing on Y and let $Y \xrightarrow{i} X$ be the inclusion map. Show that $\mathcal{O}_Y = i^*(\mathcal{O}_X/\mathcal{I}_Y)$.

Exercise 31: (*central projection*) Let $p = \langle 0, \ldots, 0, 1 \rangle \in \mathbb{P}_{n+1}(k) = \mathbb{P}_{n+1}$.

(a)Show that the map $\mathbb{P}_{n+1} \setminus \{p\} \xrightarrow{\pi} \mathbb{P}_n$, $\langle x_0, \ldots, x_{n+1} \rangle \mapsto \langle x_0, \ldots, x_n \rangle$ is well-defined and is a surjective morphism.

(b)The fibres of π are isomorphic to k as algebraic varieties.

(c) For any standard open affine set (chart) $U_i \subset \mathbb{P}_n$ there is an "obvious" isomorphism (of algebraic varieties)

$$\pi^{-1}(U_i) \xrightarrow{\tau_i} U_i \times k,$$

such that $p_i \circ \tau_i = \pi|_{\pi^{-1}(U_i)}$, where p_i is the first projection $U_i \times k \to U_i$.

(d) Calculate the map $\tau_i \circ \tau_j^{-1} : U_{ij} \times k \to U_{ij} \times k$, where $U_{ij} = U_i \cap U_j$.

(e) Let $\mathbb{P}_n \subset \mathbb{P}_{n+1}$ be embedded via $\langle x_0, \ldots, x_n \rangle \mapsto \langle x_0, \ldots, x_n, 0 \rangle$. Show that the fibres of π are the lines connecting the points of $\mathbb{P}_n$ with p (with p deleted). Draw a picture of this situation.

Exercise 32: Let X and Y be algebraic varieties over k.

1) Let $X' \subset X$ and $Y' \subset Y$ be closed (closed subvarieties). Show that $X' \times Y' \subset X \times Y$ is a closed subvariety as well.

2) Let X and Y be irreducible. Show that $X \times Y$ is irreducible as well.

Exercise 33: Let $\mathbb{P}_m \times \mathbb{P}_n \xrightarrow{h} \mathbb{P}_k$ be given by

$$(\langle x \rangle, \langle y \rangle) \to \langle f_0(x)g_0(y), \ldots, f_k(x)g_k(y) \rangle$$

such that f_i and g_i are homogeneous polynomials of grad $\deg f_i = a$ and $\deg g_i = b$ respectively, $h_i(x,y) = f_i(x)g_i(y)$ have no common zero for $x \neq 0 \neq y$. Show that h is a morphism of varieties.

Exercise 34: Let $\mathbb{P}_1 \times \mathbb{P}_1 \xrightarrow{f} \mathbb{P}_3$ be given by

$$(\langle x_0, x_1 \rangle, \langle y_0, y_1 \rangle) \mapsto \langle x_0 y_0, x_0 y_1, x_1 y_0, x_1 y_1 \rangle.$$

(1) Show that it is an example of a morphism as in Exercise 33.

(2) Show that f is injective.

(3) Show that $Q = f(\mathbb{P}_1 \times \mathbb{P}_1)$ is a surface of degree 2 and find its equation.

(4) Show that $\mathbb{P}_1 \times \mathbb{P}_1 \xrightarrow{f} Q$ is an isomorphism.

(5) Show that for any point $q \in Q$ there are exactly 2 projective lines (1-dimensional projective subspaces of $\mathbb{P}_3$) $L_1, L_2 \subset Q$ passing through q.

(6) Interpret these lines under the isomorphism from (4).

Exercise 35: Write down notes for a talk on direct and inverse images of sheaves following the Kurzskript “geringte Räume”.

Exercise 36: Let $\mathbb{F} = \mathrm{Bl}_{p_0}(\mathbb{P}_2)$ be the blow up of $\mathbb{P}_2$ at $p_0 = \langle 1, 0, 0 \rangle$,

$$\mathbb{F} := \{(\langle x_0, x_1, x_2 \rangle, \langle u_1, u_2 \rangle) \in \mathbb{P}_2 \times \mathbb{P}_1 \mid x_1 u_2 - x_2 u_1 = 0\},$$

and let $\mathbb{P}_2 \xleftarrow{\sigma} \mathbb{F} \xrightarrow{\pi} \mathbb{P}_1$ be the two morphisms resulting from the construction. Verify the following statements.

(0) $E = \sigma^{-1}(p_0)$ is isomorphic to $\mathbb{P}_1$.

(1) Each fibre of π is isomorphic to $\mathbb{P}_1$.

(2) Let $U_1, U_1 \subset \mathbb{P}_1$ be the standard charts of $\mathbb{P}_1$. Then $\pi^{-1}(U_i) \cong U_i \times \mathbb{P}_1$ via an “easy” isomorphism α_i compatible with π and pr_1. (To find α_i, consider the inverse image under σ of aline in $\mathbb{P}_2$ through p).

(3) Calculate the transformation $U_{12} \times \mathbb{P}_1 \xrightarrow{\alpha_1^{-1}} \pi^{-1}(U_{12}) \xrightarrow{\alpha_2} U_{12} \times \mathbb{P}_1$ in terms of coordinates. Here $U_{12} = U_1 \cap U_2$.

(4) Let $\mathbb{P}_1 \subset \mathbb{F}$ be embedded by $\langle u_1, u_2 \rangle \mapsto (\langle 0, u_1, u_2 \rangle, \langle u_1, u_2 \rangle)$. Draw a picture of the fibration π.

Exercise 37: Let $Q \subset \mathbb{P}_3$ be the quadric

$$Q = \{\langle x \rangle \mid x_1^2 - x_0 x_2 = 0\}$$

and let

$$C = \{\langle x \rangle \in Q \mid x_3 = 0\}.$$

Let $a = \langle 0, 0, 0, 1 \rangle$, and let

$$\mathbb{P}_3 \setminus \{a\} \xrightarrow{p} P \cong \mathbb{P}_2$$

be the central projection, where P is the plane defined by the equation $x_3 = 0$. Let $\widetilde{\mathbb{P}}_3 \subset \mathbb{P}_3 \times \mathbb{P}_2$ be the blow up of $\mathbb{P}_3$ at the point a with exceptional divisor E.

(1) Verify that Q is the union of lines connecting a with the points of C, and $C = p(Q \setminus \{a\})$.

(2) Let $\tilde{Q}$ be the proper transform of Q under $\widetilde{\mathbb{P}}_3 \xrightarrow{\sigma} \mathbb{P}_3$. Show that $\tilde{Q} = (Q \times C) \cap \widetilde{\mathbb{P}}_3$ (use the inclusion $\widetilde{\mathbb{P}}_3 \subset \mathbb{P}_3 \times \mathbb{P}_2$ and the identification $P \cong \mathbb{P}_2$).

(3) Let $\tilde{C} = E \cap \tilde{Q}$. Consider the projection $\pi : \tilde{\mathbb{P}}_3 \to \mathbb{P}_2$ (the composition $\widetilde{\mathbb{P}}_3 \subset \mathbb{P}_3 \times \mathbb{P}_2 \xrightarrow{\mathrm{pr}_2} \mathbb{P}_2$). Consider C as a subset in $\mathbb{P}_2$ via the identification $P \cong \mathbb{P}_2$. Show that $\pi|_{\tilde{C}}$ induces an isomorphism $\tilde{C} \xrightarrow{\cong} C$ and $\tilde{Q} = \tilde{C} \cup \sigma^{-1}(Q \setminus \{a\})$.

(4) Show that there is an isomorphism $\mathbb{P}_1 \times \mathbb{P}_1 \to \tilde{Q}$.
Hint: Describe the "Veronese" isomorphism $\mathbb{P}_1 \cong C \cong \tilde{C}$ and then an isomorphism $\mathbb{P}_1 \to \tilde{L}$ for the proper transform $\tilde{L}$ of any projective line of the cone Q.

Exercise 38: Let $X \subset \mathbb{P}_n(k)$ be a closed irreducible algebraic subvariety and let $I_h(X) = I(\tilde{X})$ be the ideal in $S = k[z_0, \ldots, z_n]$ of the cone $\tilde{X} \subset k^{n+1}$ of X. Let $A_h(X) = S/I_h(X)$.

(1) Show that $I_h(X)$ is a homogeneous ideal, and $A_h(X)$ becomes a graded k-algebra, called the homogeneous coordinate ring of X.

(2) Verify that $A_h(X)$ has no zero divisors.

(3) Prove that the field $\mathcal{R}(X)$ is isomorphic to the field

$$Q_h A_h(X) := \left\{ \frac{f}{g} \mid f, g \in A_h(X), g \neq 0, \deg(f) = \deg(g) \right\}.$$

Exercise 39: (Joins) Let $X \subset \mathbb{P}_n$ and $Y \subset \mathbb{P}_n$ be irreducible closed projective subvarieties with homogeneous ideals $\mathfrak{a} \subset k[x_0, \ldots, x_n]$, $\mathfrak{b} \subset k[y_0, \ldots, y_n]$. Let $\mathbb{P}_n \xrightarrow{\alpha} \mathbb{P}_{2n+1}$ and

$\mathbb{P}_n \xrightarrow{\beta} \mathbb{P}_{2n+1}$ be the embeddings

$$\langle x_0, \dots, x_n \rangle \mapsto \langle x_0, \dots, x_n, 0, \dots, 0 \rangle \text{ and } \langle y_0, \dots, y_n \rangle \mapsto \langle 0, \dots, 0, y_0, \dots, y_n \rangle$$

respectively. Let $J(X, Y)$ be the union of all lines in $\mathbb{P}_{2n+1}$ connecting a point in $\alpha(X)$ with a point in $\beta(X)$.

(1) Show that $J(X, Y)$ is the projective variety defined by the ideal generated by $\mathfrak{a}$ and $\mathfrak{b}$ in $k[x_0, \dots, x_n, y_0, \dots, y_n]$.

(1′) Let $\tilde{X} \subset k^{n+1}$ and $\tilde{Y} \subset k^{n+1}$ be the cones over X and Y respectively. Show that $\tilde{X} \times \tilde{Y}$ is the cone over $J(X, Y)$.

(2) Show that $\dim J(X, Y) = \dim X + \dim Y + 1$ and that $J(X, Y)$ is irreducible.

(3) Let $\mathbb{P}_n \xrightarrow{\delta} \mathbb{P}_{2n+1}$ be the morphism $\langle x \rangle \mapsto \langle x, x \rangle$. Show that $X \cap Y = \delta^{-1} J(X, Y)$.

(4) Let $Z = \{\langle x, y \rangle \in \mathbb{P}_{2n+1} \mid x = y\} = \delta(\mathbb{P}_n)$, let the map $\mathbb{P}_{2n+1} \setminus Z \xrightarrow{\pi} \mathbb{P}_n$ be given by $\langle x, y \rangle \mapsto \langle x - y \rangle$, and let $X \vee Y$ be the closure of $\pi(J(X, Y) \setminus Z)$. Show that

$$\dim X + \dim Y - \dim X \cap Y \leqslant \dim X \vee Y \leqslant \dim J(X, Y).$$